

ON THE ROOTS OF THE EQUATION $Z'(t) = 0$

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ABSTRACT. We have proved in this paper that the Lindelöf hypothesis generates essential contraction of distances between consecutive odd-order zeros of the function $Z'(t)$. This paper is the translation of the paper [11] into the English except part 8 that we added in order to point out the I. M. Vinogradov' scepticism on possibilities of the method of trigonometric sums.

1. THE RESULT

Let (see [5], pp. 79, 329)

$$(1.1) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \\ \vartheta(t) &= -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + \frac{1}{2}it\right) = \\ &= \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \mathcal{O}\left(\frac{1}{t}\right). \end{aligned}$$

In this paper we obtain a new consequence of the Lindelöf hypothesis. Namely, the function $Z(t)$ oscillates in the interval

$$(T, T + T^\tau)$$

where τ is arbitrary small positive fixed number. This implies that the number of maxima and minima of the function $Z(t)$ is unbounded (as $T \rightarrow \infty$).

Here is the survey of our results. Let

$$S(a, b) = \sum_{0 < a \leq n < b \leq 2a} n^{it}, \quad b \leq \sqrt{\frac{t}{2\pi}}$$

be the elementary trigonometric sum (see [8], p. 31). The following theorem holds true.

Theorem. If

$$(1.2) \quad |S(a, b)| < A(\Delta) \sqrt{at}^\Delta, \quad 0 < \Delta < \frac{1}{6}$$

then there is a root of odd order of the equation

$$(1.3) \quad Z'(t) = 0$$

in the interval

$$(1.4) \quad (T, T + T^\Delta \psi(T))$$

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where $\psi(T)$ is an arbitrarily slowly increasing function unbounded from above as $T \rightarrow \infty$ (as for example $\psi(T) = \ln \ln \dots \ln T$).

Remark 1. Any root of the odd order of the equation (1.3) is the point of local maximum or local minimum of the function $Z(t)$.

For example, in the case (see [6])

$$(1.5) \quad \Delta = \frac{35}{216} + \epsilon, \quad \epsilon > 0$$

we have the following

Corollary 1. The interval

$$(1.6) \quad (T, T + T^{35/216+\epsilon})$$

contains a point of maximum or minimum of the function $Z(t)$.

On the Lindelöf hypothesis we have (see [4])

$$(1.7) \quad |S(a, b)| < A(\epsilon) \sqrt{at}^\epsilon,$$

i. e. we have the following

Corollary 2. On the Lindelöf hypothesis the interval

$$(1.8) \quad (T, T + T^\epsilon \psi(T))$$

contains a point of maximum or minimum of the function $Z(t)$.

Remark 2. Thus the Lindelöf hypothesis gives an 100% improvement of the exponent $\frac{35}{316}$ in (1.6). Simultaneously, we have an 100% improvement of all incoming exponents Δ of the kind (1.2), (1.5).

Next, let $N'_0(T)$ denote the number of the zeros of the equation (1.3) for $t \in (0, T]$. We have the following

Corollary 3. On the Lindelöf hypothesis we have the estimate

$$(1.9) \quad N'_0(T + T^\tau) - N'_0(T) > A(\tau, \epsilon) T^{\tau-\epsilon}, \quad 0 < \epsilon < \tau$$

where τ is an arbitrary small fixed number.

There are the following reasons to study the roots of the equation (1.3):

- (a) the truth of the Riemann hypothesis itself is connected with the question on distribution of the roots of the equation (1.3) (see, for example, [7], p. 34, Corollary 3),
- (b) if the Riemann hypothesis is true then the roots of the equation (1.3) are connected with the question: are there multiple (≥ 2) zeros of the function $\zeta(\frac{1}{2} + it)$?

2. THE MAIN FORMULAE

The proof of our Theorem lies on the following

Formula 1.

$$(2.1) \quad Z'(t) = -2 \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{\sqrt{n}} (\vartheta' - \ln n) \sin(\vartheta - t \ln n) + \mathcal{O}(t^{-1/4} \ln t).$$

The proof of this formula is situated in the parts 3 and 4 of this text. Next, in the part 5, the formula (2.1) is transformed into the following

Formula 2.

$$(2.2) \quad \begin{aligned} Z'(t) &= -2 \sum_{n < P_0} \frac{1}{\sqrt{n}} \ln \frac{P_0}{n} \sin(\vartheta - t \ln n) + \mathcal{O}(T^{-1/4} \ln T), \\ t &\in [T, T+H], \quad H \in (0, \sqrt[4]{T}], \quad P_0 = \sqrt{\frac{T}{2\pi}}. \end{aligned}$$

Let the sequence $\{\tilde{t}_\nu\}$ be defined by the formula

$$(2.3) \quad \vartheta(\tilde{t}_\nu) = \pi\nu + \frac{\pi}{2}, \quad \nu \in \mathbb{N}.$$

We obtain in the part 6 the following statement.

Lemma 1. *From (1.2) the estimate*

$$(2.4) \quad \sum_{T \leq \tilde{t}_\nu \leq T+H} Z'(\tilde{t}_\nu) = \mathcal{O}(T^\Delta \ln^2 T)$$

follows.

Next, in the part 7, we obtain

Lemma 2. *From (1.2) the formula*

$$(2.5) \quad \sum_{T \leq \tilde{t}_\nu \leq T+H} (-1)^\nu Z'(\tilde{t}_\nu) = -\frac{1}{2\pi} H \ln^2 \frac{T}{2\pi} + \mathcal{O}(T^\Delta \ln^2 T)$$

follows.

Remark 3. If

$$T^\Delta = o(H)$$

then (2.5) is the asymptotic formula, i. e. if, for example,

$$(2.6) \quad H = T^\Delta \psi(T).$$

Finally, from (2.4) and (2.5) we obtain the following

Lemma 3.

$$(2.7) \quad \begin{aligned} \sum_{T \leq \tilde{t}_{2\nu} \leq T+H} Z'(\tilde{t}_{2\nu}) &= -\frac{1}{4\pi} H \ln^2 \frac{T}{2\pi} + \mathcal{O}(T^\Delta \ln^2 T), \\ \sum_{T \leq \tilde{t}_{2\nu+1} \leq T+H} Z'(\tilde{t}_{2\nu+1}) &= \frac{1}{4\pi} H \ln^2 \frac{T}{2\pi} + \mathcal{O}(T^\Delta \ln^2 T). \end{aligned}$$

In the case (2.6) the assertion of the Theorem follows from (2.7).

3. PROOF OF THE FORMULA 1 (THE FIRST PART)

3.1. We use the formula (see [15], p. 72, $x = y = \sqrt{\frac{t}{2\pi}}$, $\eta = \sqrt{2\pi t}$, $m = [x]$)

$$(3.1) \quad \begin{aligned} \zeta(s) &= \sum_{n=1}^m \frac{1}{n^s} + \chi(s) \sum_{n=1}^m \frac{1}{n^{1-s}} + \\ &+ \frac{e^{-i\pi s} \Gamma(1-s)}{2\pi i} \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) \frac{w^{s-1} e^{-mw}}{e^w - 1} dw, \quad s = \sigma + it \end{aligned}$$

where C_1, \dots, C_4 are the segments binding the following points in the w -plane

$$(3.2) \quad \begin{aligned} & \infty + i\eta(1+c), \quad c\eta + i\eta(1+c), \\ & c\eta + i\eta(1+c), \quad -c\eta + i\eta(1-c), \\ & -c\eta + i\eta(1-c), \quad -c\eta - i(2m+1)\pi, \\ & -c\eta - i(2m+1)\pi, \quad \infty - i(2m+1)\pi, \end{aligned}$$

correspondingly, and

$$0 < c \leq \frac{1}{2}.$$

Putting

$$s = \frac{1}{2} + it$$

into (3.1) and multiplying the last by $e^{i\vartheta(t)}$, (comp. [15], p. 79) we obtain

$$(3.3) \quad \begin{aligned} Z(t) &= e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right) = \\ &= 2 \sum_{n \leq \alpha(t)} \frac{1}{\sqrt{n}} \cos(\vartheta - t \ln n) - \frac{1}{2\pi} e^{\pi t + i\vartheta} \Gamma\left(\frac{1}{2} - it\right) W(t) \end{aligned}$$

where

$$(3.4) \quad W(t) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) \psi(t, w) dw = \int_{C(t)} \psi(t, w) dw,$$

and

$$(3.5) \quad \psi(t, w) = \frac{w^{-1/2+it} e^{-mw}}{e^w - 1}, \quad \alpha(t) = \sqrt{\frac{t}{2\pi}}.$$

3.2. Putting

$$(3.6) \quad \Phi(t) = \sum_{n \leq \alpha(t)} \frac{1}{\sqrt{n}} \cos(\vartheta - t \ln n)$$

we obtain (let, for example, $\delta > 0$)

$$(3.7) \quad \begin{aligned} & \Phi(t + \delta) - \Phi(t) = \\ &= \sum_{n \leq \alpha(t)} \frac{1}{\sqrt{n}} [\cos(\vartheta(t + \delta) - (t + \delta) \ln n) - \cos(\vartheta(t) - t \ln n)] + \\ &+ \sum_{\alpha(t) < n \leq \alpha(t + \delta)} \frac{1}{\sqrt{n}} \cos(\vartheta(t + \delta) - (t + \delta) \ln n) = \Sigma_1 + \Sigma_2. \end{aligned}$$

Since

$$(3.8) \quad \alpha(t + \delta) - \alpha(t) = \mathcal{O}\left(\frac{\delta}{\sqrt{t}}\right),$$

then we can choose δ in such a way that the interval

$$(\alpha(t), \alpha(t + \delta)]$$

does not contain a natural number. Consequently,

$$(3.9) \quad \Sigma_2 = 0.$$

3.3. Next, we have (see (3.4))

$$\begin{aligned}
 W(t + \delta) - W(t) &= \int_{C(t+\delta)} \psi(t + \delta, w) dw - \int_{C(t)} \psi(t, w) dw = \\
 &= \int_{C(t+\delta)} \psi(t + \delta, w) dw - \int_{C(t)} \psi(t + \delta, w) dw + \\
 (3.10) \quad &+ \int_{C(t)} [\psi(t + \delta, w) - \psi(t, w)] dw = \\
 &= \int_{C(t+\delta) \cup \{-C(t)\}} \psi(t + \delta, w) dw + \int_{C(t)} [\psi(t + \delta, w) - \psi(t, w)] dw = \\
 &= \int_{C(t)} [\psi(t + \delta, w) - \psi(t, w)] dw
 \end{aligned}$$

by the Cauchy theorem, since

$$\psi(t + \delta, w)$$

is the analytic function with respect to the variable w (and δ is arbitrarily small) in the region bounded by the contour $C(t + \delta) \cup \{-C(t)\}$. Consequently,

$$(3.11) \quad W'(t) = \int_{C(t)} \frac{\partial \psi(t, w)}{\partial t} dw.$$

3.4. Hence, from (3.3), by (3.2), (3.3), we obtain

$$\begin{aligned}
 Z'(t) &= -2 \sum_{n \leq \alpha(t)} \frac{1}{\sqrt{n}} (\vartheta' - \ln n) \sin(\vartheta - t \ln n) + \\
 (3.12) \quad &+ \frac{1}{2\pi i} \left\{ \vartheta' - \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} - it \right) - \pi i \right\} e^{i\vartheta + \pi t} \Gamma \left(\frac{1}{2} - it \right) W(t) - \\
 &- \frac{1}{2\pi} e^{i\vartheta + \pi t} \Gamma \left(\frac{1}{2} - it \right) W'(t) = \\
 &= Z_1(t) + Z_2(t) + Z_3(t).
 \end{aligned}$$

We can apply the analysis from [15], pp. 71-74 in the case of the function

$$e^{i\vartheta} e^{\pi t - i\pi/2} \Gamma \left(\frac{1}{2} - it \right) W(t)$$

and we obtain the estimate

$$\mathcal{O}(t^{-1/4}).$$

Furthermore, by [15], pp. 25, 221, we have

$$(3.13) \quad \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} - it \right) = \mathcal{O}(\ln t), \quad \vartheta'(t) = \mathcal{O}(\ln t),$$

and then we obtain

$$(3.14) \quad Z_2(t) = \mathcal{O}(t^{-1/4} \ln t).$$

4. PROOF OF THE FORMULA 1 (THE SECOND PART)

Since by (3.5) we have

$$(4.1) \quad \frac{\partial \psi(t, w)}{\partial t} = i \ln w \frac{w^{-1/2+it} e^{-mw}}{e^w - 1},$$

then we obtain the estimate of $Z_3(t)$ (see (3.12)) by the method [15], pp. 71-74 (we must to consider the influence of the factor $\ln w$).

4.1. We have on the contour C_4 (see [15], p. 72)

$$(4.2) \quad \begin{aligned} w &= u + i(2m+1)\pi, \\ \ln w &= \begin{cases} \mathcal{O}(\ln u) & , \quad u > m, \\ \mathcal{O}(\ln t) & , \quad u \in (-c\eta, m], \end{cases} \end{aligned}$$

and, consequently,

$$(4.3) \quad \int_{C_4} = \mathcal{O} \left\{ \eta^{-1/2} e^{-5\pi t/4} \int_{-c\eta}^{\infty} e^{-mu} |\ln w| du \right\} = \mathcal{O} \left\{ e^{t(c-5\pi/4)} \ln t \right\}.$$

4.2. On C_3 we have

$$|\ln w| < A \ln t,$$

then (comp. [15], p. 72)

$$(4.4) \quad \int_{C_3} = \mathcal{O} \left\{ e^{-t(\pi/2+A)} \ln t \right\}.$$

4.3. On C_1 we have

$$(4.5) \quad |\ln w| = \begin{cases} \mathcal{O}(\ln t) & , \quad 0 < u \leq \pi\eta, \\ \mathcal{O}(\ln u) & , \quad u > \pi\eta, \end{cases}$$

then (comp. [15], p. 72)

$$(4.6) \quad \begin{aligned} \int_{C_1} &= \mathcal{O} \left\{ \eta^{-1/2} \ln t \int_0^{\pi\eta} e^{-(\pi/2+A)t} du \right\} + \mathcal{O} \left\{ \eta^{-1/2} \int_{\pi\eta}^{\infty} e^{-xu} \ln u du \right\} = \\ &= \mathcal{O} \left\{ \eta^{1/2} e^{-(\pi/2+A)t} \ln t \right\}, \quad \eta = 2\pi x. \end{aligned}$$

4.4. Since on C_2

$$w = i\eta + \lambda e^{i\pi/4}, \quad |\lambda| < \sqrt{2c\eta}, \quad \ln w = \mathcal{O}(\ln t),$$

then we obtain for the corresponding parts of the integral the following estimate (comp. [15], pp. 73, 74)

$$(4.7) \quad \begin{aligned} \mathcal{O}(\eta^\sigma t^{-1/2} e^{-\pi t/2}) &\rightarrow \mathcal{O}(\eta^{1/2} t^{-1/2} e^{-\pi t/2} \ln t), \\ \mathcal{O}(\eta^{\sigma-1} e^{-\pi t/2}) &\rightarrow \mathcal{O}(\eta^{-1/2} e^{-\pi t/2} \ln t). \end{aligned}$$

4.5. Since (comp. [15], p. 74)

$$(4.8) \quad e^{i\vartheta+\pi t} \Gamma\left(\frac{1}{2} - it\right) = \mathcal{O}(e^{\pi t/2}),$$

then by (3.12), (4.3), (4.4), (4.6), (4.7) we obtain the estimate

$$(4.9) \quad Z_3(t) = \mathcal{O}(t^{-1/4} \ln t).$$

Finally, from (3.12) by (3.14), (4.9) we obtain (2.1).

5. PROOF OF THE FORMULA 2

If

$$H \in (0, \sqrt[4]{T}], \quad t \in [T, T+H],$$

then

$$(5.1) \quad \alpha(T+H) - \alpha(T) = \mathcal{O}\left(\frac{H}{\sqrt{T}}\right) = \mathcal{O}(T^{-1/4}),$$

and, by (3.13),

$$(5.2) \quad \sum_{\alpha(T) \leq n \leq \alpha(T+H)} \frac{1}{\sqrt{n}} (\vartheta' - \ln n) \sin(\vartheta - t \ln n) = \mathcal{O}(T^{-1/4} \ln T).$$

Consequently, we have from (2.1) by (5.2)

$$(5.3) \quad Z'(t) = -2 \sum_{n < P_0} \frac{1}{\sqrt{n}} (\vartheta' - \ln n) \sin(\vartheta - t \ln n) + \mathcal{O}(T^{-1/4} \ln T)$$

where

$$P_0 = \alpha(T) = \sqrt{\frac{T}{2\pi}}.$$

Next, by the formulae (see [15], pp. 221)

$$(5.4) \quad \vartheta'(t) = \frac{1}{2} \ln \frac{t}{2\pi} + \mathcal{O}\left(\frac{1}{t}\right), \quad \vartheta''(t) \sim \frac{1}{2t},$$

we have

$$(5.5) \quad \vartheta'(t) = \vartheta'(T) + \mathcal{O}\left(\frac{H}{T}\right) = \ln P_0 + \mathcal{O}(T^{-3/4}).$$

Since

$$(5.6) \quad \sum_{n < P_0} \frac{1}{\sqrt{n}} \mathcal{O}(T^{-3/4}) = \mathcal{O}(T^{-1/2}),$$

then we obtain from (5.3) by (5.5), (5.6)

$$Z'(t) = -2 \sum_{n < P_0} \frac{1}{\sqrt{n}} \ln \frac{P_0}{n} \sin(\vartheta - t \ln n) + \mathcal{O}(T^{-1/4} \ln T),$$

i. e. the formula (2.2).

6. PROOF OF THE LEMMA 1

6.1. Since (see (2.3))

$$(6.1) \quad \sin\{\vartheta(\tilde{t}_\nu)\} = (-1)^\nu, \quad \cos\{\vartheta(\tilde{t}_\nu)\} = 0,$$

then we obtain from (2.2) the formula

$$(6.2) \quad Z'(\tilde{t}_\nu) = 2(-1)^{\nu+1} \sum_{n < P_0} \frac{1}{\sqrt{n}} \ln \frac{P_0}{n} \cos\{\tilde{t}_\nu \ln n\} + \mathcal{O}(T^{-1/4} \ln T),$$

and for the distance

$$\tilde{t}_{\nu+1} - \tilde{t}_\nu$$

we have the formula

$$(6.3) \quad \tilde{t}_{\nu+1} - \tilde{t}_\nu = \frac{2\pi}{\ln \frac{T}{2\pi}} + \mathcal{O}\left(\frac{H}{T \ln^2 T}\right) = \frac{\pi}{\ln P_0} + \mathcal{O}\left(\frac{T^{-3/4}}{\ln T}\right)$$

(similarly to [14], p. 102; [8], (42)).

6.2. Next, we have the following remarks:

- (a) the formula (6.3) for the difference $\tilde{t}_{\nu+1} - \tilde{t}_\nu$ is asymptotically equal to the formula for the difference $t_{\nu+1} - t_\nu$ (comp. [8], (42)),
- (b) the formula for $Z'(\tilde{t}_\nu)$ (see (6.2)) differs from the formula (see [15], p. 221)

$$(6.4) \quad Z(t_\nu) = 2(-1)^\nu \sum_{n < P_0} \frac{1}{\sqrt{n}} \cos(t_\nu \ln n) + \mathcal{O}(T^{-1/4})$$

by the factors

$$-\ln \frac{P_0}{n}, \quad \ln T$$

in the sum and in the error term, correspondingly.

6.3. It is clear by (6.2) that our method of reduction

$$\sum_{T \leq t_\nu \leq T+H} Z(t_\nu) \rightarrow W(t)$$

(see [8], (59)-(61)) is applicable also in the present case if we use the substitutions

$$\frac{1}{\sqrt{n}} \rightarrow -\frac{1}{\sqrt{n}} \ln \frac{P_0}{n}, \quad \mathcal{O}\{\dots\} \rightarrow \mathcal{O}\{\dots\} \ln T.$$

6.4. By (6.3) we have (see [8], (59)-(61))

$$(6.5) \quad \sum_{T \leq \tilde{t}_\nu \leq T+H} Z'(\tilde{t}_\nu) = -2\tilde{W}(T, H) + \mathcal{O}(\ln^2 T)$$

where ($n < P_0$)

$$(6.6) \quad \begin{aligned} \tilde{W}(T, H) &= \frac{1}{2}(-1)^{\tilde{\nu}} \sum \frac{\ln \frac{P_0}{n}}{\sqrt{n}} \cos(\varphi) + \\ &+ \frac{1}{2}(-1)^{N+\tilde{\nu}} \sum \frac{\ln \frac{P_0}{n}}{\sqrt{n}} \cos(\omega N + \varphi) + \\ &+ \frac{1}{2}(-1)^{\tilde{\nu}} \sum \frac{\ln \frac{P_0}{n} \tan \frac{\omega}{2}}{\sqrt{n}} \sin(\varphi) + \\ &+ \frac{1}{2}(-1)^{N+\tilde{\nu}+1} \sum \frac{\ln \frac{P_0}{n} \tan \frac{\omega}{2}}{\sqrt{n}} \sin(\omega N + \varphi) + \mathcal{O}(\ln^2 T) = \\ &= \tilde{W}_1 + \tilde{W}_2 + \tilde{W}_3 + \tilde{W}_4 + \mathcal{O}(\ln^2 T), \end{aligned}$$

and (see [8], (43), (50))

$$(6.7) \quad \begin{aligned} \tilde{t}_{\tilde{\nu}} &= \min_{\tilde{t}_\nu \in [T, T+H]} \{\tilde{t}_\nu\}, \quad \tilde{t}_{\tilde{\nu}+N} = \max_{\tilde{t}_\nu \in [T, T+H]} \{\tilde{t}_\nu\}, \\ \omega &= \pi \frac{\ln n}{\ln P_0}, \quad \varphi = \tilde{t}_{\tilde{\nu}} \ln n. \end{aligned}$$

6.5. Since (see [8], (70))

$$(6.8) \quad \sum_{1 \leq n < M \leq P_0} \frac{\cos \varphi}{\sqrt{n}} = \mathcal{O}(T^\Delta \ln T),$$

and the sequence

$$\left\{ \ln \frac{P_0}{n} \right\}$$

is decreasing and bounded by $A \ln T$ then we obtain by making use of the Abel's transformation of the term \tilde{W}_1 (see (6.6) and (6.8))

$$(6.9) \quad \tilde{W}_1 = \mathcal{O}(T^\Delta \ln^2 T),$$

and similarly

$$(6.10) \quad \tilde{W}_2 = \mathcal{O}(T^\Delta \ln^2 T).$$

6.6. For the sequence

$$(6.11) \quad \left\{ \ln \frac{P_0}{n} \tan \frac{\omega}{2} \right\}$$

we have (see (6.7)) that

$$(6.12) \quad \tan \frac{\omega}{2} = \tan \left(\frac{\pi}{2} - \frac{\pi}{2} \frac{\ln \frac{P_0}{n}}{\ln P_0} \right) = \cot \left(\frac{\pi}{2} \frac{\ln \frac{P_0}{n}}{\ln P_0} \right) = \cot X(n),$$

and consequently

$$(6.13) \quad \ln \frac{P_0}{n} \tan \frac{\omega}{2} = \frac{2}{\pi} \ln P_0 X(n) \cot X(n).$$

Next, the sequence

$$\{X(n) \cot X(n)\}; \quad 0 < X(n) \leq \frac{\pi}{2}, \quad 1 \leq n < P_0$$

is bounded by the value 1, and it is increasing, since

$$\frac{d}{dn} X \cot X = \frac{\pi}{2n \ln P_0} \frac{1 - \frac{\sin 2X}{2X}}{\sin^2 X} > 0, \quad n \in [1, P_0).$$

Then, using the Abel's transformation and the estimate (comp. (6.8))

$$\sum_{1 \leq n < M \leq P_0} \frac{\sin \varphi}{\sqrt{n}} = \mathcal{O}(T^\Delta \ln T),$$

we obtain (see (6.13))

$$(6.14) \quad \sum_{n < P_0} X(n) \cot X(n) \frac{\sin \varphi}{\sqrt{n}} = \mathcal{O}(T^\Delta \ln T),$$

i. e. (see (6.6), (6.14))

$$(6.15) \quad \tilde{W}_3 = \mathcal{O}(T^\Delta \ln^2 T),$$

and simultaneously

$$(6.16) \quad \tilde{W}_4 = \mathcal{O}(T^\Delta \ln^2 T).$$

Finally, from (6.5) by (6.6), (6.9), (6.10), (6.15), (6.16) we obtain (2.4).

7. PROOF OF THE LEMMA 2

Let us remind that (see [9], (26))

$$\sum_{T \leq t_\nu \leq T+H} (-1)^\nu Z(t_\nu) = \frac{1}{\pi} H \ln \frac{T}{2\pi} + \tilde{W}(T, H) + \mathcal{O}\left(\frac{H^2}{T}\right),$$

where

$$\tilde{W} = 2 \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu \leq T+H} \cos(t_\nu \ln n),$$

and (see [9], (51))

$$(7.1) \quad \tilde{W} = \mathcal{O}(T^\Delta \ln T).$$

Since (see (6.3) and [9], (23))

$$\sum_{T \leq \tilde{t}_\nu \leq T+H} 1 = \frac{1}{2\pi} \ln \frac{T}{2\pi} + \mathcal{O}(T^{-1/2}) = \frac{1}{\pi} H \ln P_0 + \mathcal{O}(T^{-1/2}),$$

then we have (see (6.2))

$$\begin{aligned} (7.2) \quad & \sum_{T \leq \tilde{t}_\nu \leq T+H} (-1)^\nu Z'(\tilde{t}_\nu) = \\ & = -2 \ln P_0 \sum_{\tilde{t}_\nu} 1 - 2 \sum_{2 \leq n < P_0} \frac{\ln \frac{P_0}{n}}{\sqrt{n}} \sum_{\tilde{t}_\nu} \cos(\tilde{t}_\nu \ln n) + \sum_{\tilde{t}_\nu} \mathcal{O}(T^{-1/4} \ln T) = \\ & = -2 \ln P_0 \left\{ \frac{1}{\pi} H \ln P_0 + \mathcal{O}(T^{-1/2}) \right\} - R(T, H) + \mathcal{O}(\ln^2 T), \end{aligned}$$

and, of course, (comp. (7.1))

$$\sum_{2 \leq n < M \leq P_0} \frac{1}{\sqrt{n}} \sum_{T \leq \tilde{t}_\nu \leq T+H} \cos(\tilde{t}_\nu \ln n) = \mathcal{O}(T^\Delta \ln T).$$

Hence, using the Abel's transformation on R , we obtain the estimate

$$(7.3) \quad R = \mathcal{O}(T^\Delta \ln^2 T).$$

Finally, we obtain (2.5) from (7.2) by (7.3).

APPENDIX A. ON I.M. VINOGRADOV' SCEPTICISM ON POSSIBILITIES OF THE METHOD OF TRIGONOMETRIC SUMS

A.1. I.M. Vinogradov analyzed in the Introduction to his monograph [16] the possibilities of the method of trigonometric sums (Weyl's sums) in the problem of estimation of the remainder term $R(N)$ in the asymptotic formula (see [16], p. 13)

$$(A.1) \quad \pi(N) - \int_2^N \frac{dx}{\ln x} = R(N)$$

where $\pi(N)$ is the prime-counting function. Vinogradov made the following remark in this:

Obviously, it is very hard to make an essential progress in solution of the problem to find the order of the R -term (willing to find $R = \mathcal{O}(N^{1-c})$, $c = 0.000001$) by making use of only some improvements of the H. Weyl's estimates and without making use of further important progresses in the theory of the zeta-function.

A.2. We will discuss in this section an analogue of the I.M. Vinogradov's scepticism in the case of estimation of the remainder term for the Hardy-Littlewood integral in the formula

$$(A.2) \quad \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right| dt - T \ln T - (2c - 1 - \ln 2\pi)T = Q(T)$$

that is an analogue to (A.1). More generally, we ask whether there is a finer representation of the Hardy-Littlewood integral than the one in the formula (A.2).

The first mathematician who applied the method of trigonometric sums to estimate $Q(T)$ was E.C. Titchmarsh (1934). He received the result (comp. [15], p. 123)

$$Q(T) = \mathcal{O}(T^{5/12+\epsilon}).$$

The crucial result in this field was obtained by Good (see [1])

$$(A.3) \quad Q(T) = \Omega(T^{1/4}), \quad T \rightarrow \infty.$$

Remark 4. It follows from (A.3) that

$$(A.4) \quad Q(T) = \mathcal{O}(T^{1/4+\epsilon}).$$

This Good's estimate still represents an unbounded and unremovable absolute error in the formula (A.2), and is the final eventuality for the method of trigonometric sums in this question.

Next, we have proved in our paper [12] (90 years after the classical Hardy-Littlewood's paper [2]) the following: there is an infinite set of the almost exact representations of the Hardy-Littlewood integral

$$(A.5) \quad \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right| dt,$$

namely

$$(A.6) \quad \int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right| dt = \varphi_1(T) \ln \varphi_1(T) + (c - \ln 2\pi)\varphi_1(T) + c_0 + \mathcal{O}\left(\frac{\ln T}{T}\right), \quad T \rightarrow \infty; \quad \varphi_1(t) = \frac{1}{2}\varphi(t),$$

(comp. [13], (9.1)), where c is the Euler's constant, c_0 is the constant from the Titchmarsh-Kober-Atkinson formula (see [15], p. 141), and $\varphi(T)$ is a solution to the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t) e^{-\frac{2}{x(T)}t} dt = \int_0^T Z^2(t) dt; \quad \mu(y) \geq 7 \ln y$$

in which each admissible function $\mu(y)$ (see [12]) generates a solution

$$y = \varphi_\mu(T) = \varphi(T).$$

Remark 5. The result (A.6) can be formulated as follows: the Jacob's ladders $\varphi_1(T)$ (comp. (A.6) and the extension in [12], p. 415; $G[\varphi(t)]$) are the asymptotic solutions of the following new transcendental equation

$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right| dt = V(T) \ln V(T) + (c - \ln 2\pi)V(T) + c_0.$$

Remark 6. In the question on a possibility to find a finer representation of the Hardy-Littlewood integral (A.5) is the following situation:

- (a) we have for the classical representation (A.2)

$$(A.7) \quad \limsup_{T \rightarrow \infty} |Q(T)| = +\infty,$$

(see (A.3), (A.4)), i. e. we have the unbounded and unremovable absolute error term for all methods which use the estimation of trigonometric sums,

- (b) in our representation (A.6) of the Hardy-Littlewood integral that is absolutely independent on the method of trigonometric sums, we have

$$(A.8) \quad \lim_{T \rightarrow \infty} Q_1(T) = 0; \quad Q_1(T) = \mathcal{O}\left(\frac{\ln T}{T}\right),$$

i. e. we have a negligible error term,

- (c) hence, the results (A.7), (A.8) confirm the validity of the analogue of I.M. Vinogradov's scepticism in the question of accuracy of the representation of the Hardy-Littlewood integral (A.5).

A.3. We give in this part also other analogue of the I.M. Vinogradov's scepticism. First of all we would like to describe our two main goals when working with the Titchmarsh' sequence

$$\{Z(t_\nu)\},$$

where $\{t_\nu\}$ is the Gram's sequence. We wanted to:

- (a) improve the knowledge about the local variant of the classical Titchmarsh' formulae (see [15], pp. 221, 222)

$$\begin{aligned} \sum_{\nu=\nu_0}^N Z(t_{2\nu}) &= 2N + \mathcal{O}(N^{3/4} \ln^{3/4} N), \\ \sum_{\nu=\nu_0}^N Z(t_{2\nu+1}) &= -2N + \mathcal{O}(N^{3/4} \ln^{3/4} N), \end{aligned}$$

- (b) prove a mean-value theorems for the function $Z(t)$ on related non-connected sets.

In order to solve the tasks (a) and (b) we have first used the method of trigonometric sums (see [3], pp. 260, 265; [5], p. 37). In the task (a) we have improved the Titchmarsh' exponent as $\frac{3}{4} \rightarrow \frac{1}{6}$, i. e. we improved it by 77.7%. In the task (b) we obtained a new class of mean-value theorems (see [10]) corresponding to the exponent $\frac{1}{6}$

$$\begin{aligned} \frac{1}{m\{G_1(x)\}} \int_{G_1(x)} Z(t) dt &\sim 2 \frac{\sin x}{x}, \\ \frac{1}{m\{G_2(y)\}} \int_{G_2(y)} Z(t) dt &\sim -2 \frac{\sin y}{y}, \\ 0 < x, y &\leq \frac{\pi}{2}, \quad T \rightarrow \infty. \end{aligned}$$

What concerns the task (a) – we have essentially improved the Hardy-Littlewood exponent $\frac{1}{4}$ (since 1918) to the value $\frac{1}{6}$ (the problem of estimation of the distance between neighboring zeros of the function $\zeta\left(\frac{1}{2} + it\right)$, see [2], p. 125, 177-184).

Let us follow the sequence of improvements of the mentioned Hardy-Littlewood exponent

- Moser – 33% improvement of the Hardy-Littlewood exponent $\frac{1}{4}$,
- Karatsuba – 6.25% improvement of the exponent $\frac{1}{6}$,
- Ivič – 0.19% improvement of the Karatsuba exponent.

Remark 7. The sequence of improvements of the kind 6.25%, 0.19%, ... shows that the scepticism of I.M. Vinogradov takes place also in the possibility of successful application of the method of trigonometric sums in the problem of crucial improvement of the exponent $\frac{1}{6}$.

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